Intrinsically anomalous self-similarity of randomly folded matter

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We found that randomly folded thin sheets exhibit unconventional scale invariance, which we termed as an intrinsically anomalous self-similarity, because the self-similarity of the folded configurations and of the set of folded sheets are characterized by different fractal dimensions. Besides, we found that self-avoidance does not affect the scaling properties of folded patterns, because the self-intersections of sheets with finite bending rigidity are restricted by the finite size of crumpling creases, rather than by the condition of self-avoidance. Accordingly, the local fractal dimension of folding structures is found to be universal ($D_l=2.64\pm0.05$) and close to expected for a randomly folded phantom sheet with finite bending rigidity. At the same time, self-avoidance is found to play an important role in the scaling properties of the set of randomly folded sheets of different sizes, characterized by the material-dependent global fractal dimension $D<D_l$. So intrinsically anomalous self-similarity is expected to be an essential feature of randomly folded thin matter.

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Randomly folded matter is ubiquitous in nature [1–3]. Over the years, considerable amounts of experimental and theoretical research have been performed to understand the folding geometry and its effect on the mechanical behavior of randomly folded matter [4–11]. It was found that almost any thin material crumples in such a way that the folding energy is concentrated in the network of narrow ridges that meet in the pointlike vertices [5–11]. The ridge length distribution is found to conform to the log-normal or γ distribution [2,5–8,10] with the mean Δ proportional to the diameter of the randomly folded state R [9,10]. The balance of bending and stretching energy stored in the folded creases determines the scaling properties of the folded state as a function of the sheet size L, thickness $h \ll L$, two-dimensional Young's modulus Y, and confinement force F [7,11]. Specifically, numerical simulations of randomly folded elastic sheets performed in [7] suggest that

$$\frac{R}{h} \propto \left(\frac{L}{h}\right)^{2/D}\left(\frac{F}{Yh}\right)^{-\delta}, \qquad (1)$$

where D is the mass fractal dimension and δ is the force scaling exponent. Hence, the set of randomly folded sheets of different sizes is expected to obey a fractal law $M \propto L^2 \propto R^D$ if all sheets are folded under the same confinement force—i.e., $F=\text{const}$ [7–12]. Furthermore, numerical simulations with a coarse-grained model of triangulated self-avoiding surfaces with bending and stretching elasticity [7] suggest different scaling exponents for the phantom ($D=8/3$ and $\delta=3/8$) and self-avoiding ($D=2.3$ and $\delta=1/4$) sheets with the finite bending rigidity [13]. An experimental study performed with thin aluminum sheets suggests that the scaling properties of the set of randomly folded predominantly plastic sheets of different size and thickness are determined by the effect of sheet self-avoidance [11], whereas the mass fractal dimension of randomly folded elastoplastic sheets is found to be material dependent, due to the strain relaxation after the folding force is withdrawn [10].

We note that in all experimental works (e.g., [10,11,14,15]) the fractal dimension of folded sheets was determined using the scaling relation (1), and so D characterizes the self-similarity of the set of randomly folded sheets of different sizes. At the same time, the internal structure of the folded state is also expected to possess scale invariance (self-similarity) characterized by the same mass fractal dimension D [12,14–16], as is commonly observed for statistically self-similar fractals [17]. However, this property was never verified for the randomly folded matter. Accordingly, in this work we study the scaling properties of the internal structure of randomly folded elastoplastic sheets

Specifically, we used the square sheets of three different kinds of paper (carbon, biblia, and albanene), early used in Ref. [10]. The sheet size was varied from $L_0=2$ to 100 cm with the relation $L=\lambda L_0$ for the scaling factor $\lambda=1$, 2, 4, 8, 16, 32, and 50. At least 30 sheets of each size of each paper were crumpled by hand into approximately spherical balls [see Figs. 1(a) and 1(d)]. To reduce the uncertainties caused by variations in the squeezing force and strain relaxation after the folding force is withdrawn, all measurements in this work were performed 10 days after the balls were folded, when no changes in the ball dimensions were observed (see for details Ref. [10]).

The global fractal dimension D was determined using the scaling relation (1) for sets of randomly folded sheets of different sizes of each paper [see Fig. 2(a)]. The ensemble averaged diameter of balls folded from sheets of size L is defined as $R(L)=\langle \overline{R_j(L)} \rangle$, where the brackets denote average over $N=30$ balls of diameter $\overline{R_j(L,h)}=(1/n)\Sigma_i^n R_i$ and R_i are diameters measured along $n=15$ directions taken at random. We noted that the values of D measured in this work coincide with those reported in [10]. We also studied the number of intersections of a sheet with the silk string crossing over the ball along its diameter [see Figs. 1(a)–1(c)] as a function of the sheet size and the confinement ratio $K=L/R \propto R^{(D-2)/2}$. We found that the ensemble-averaged number of intersections is equal to $N=a(L/R)^2=aK^2$ [see Fig. 2(b)], where a is the material-dependent constant, and so $N \propto R^{D-2}$, where $D-2$ coincides with the fractal dimension of the disconnected set of points belonging to the intersection of the D-dimensional fractal ball with the one-dimensional

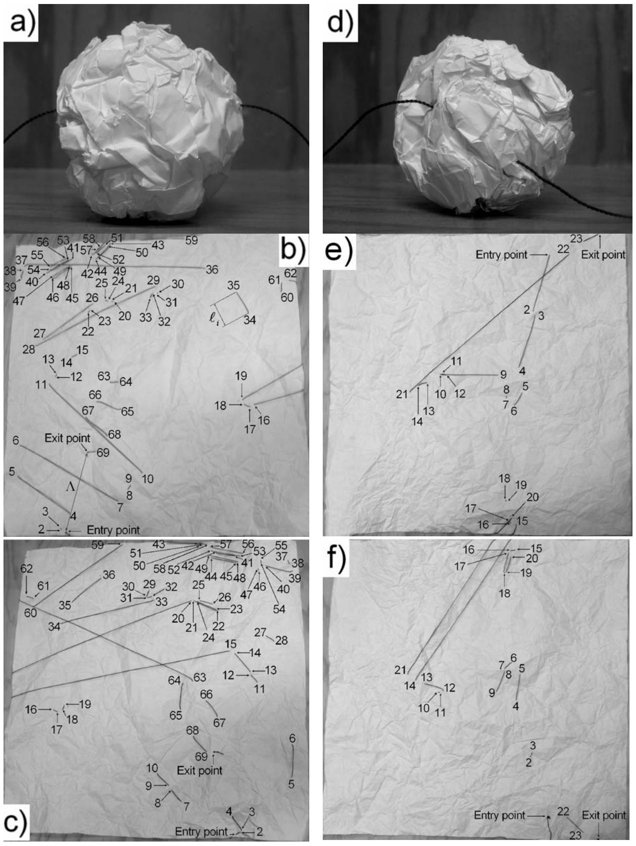

FIG. 1. Folded balls crossed over by silk string along their (a) diameter and chord (d) and (b),(c) and (d),(e) their unfolded states, respectively. The crossing point numbering corresponds to the string path.

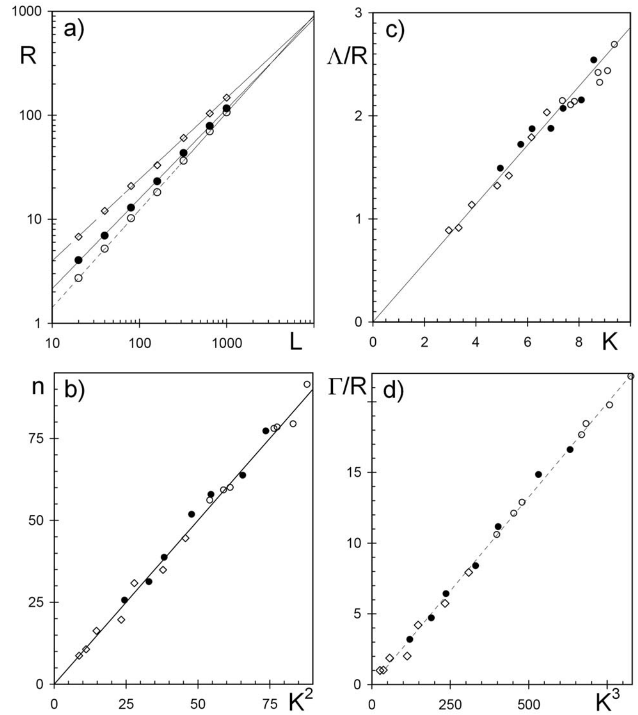

FIG. 2. Log-log plots of (a) R (mm) versus L (mm): the slopes of straight lines are equal to 0.94 (dashed line), 0.86 (solid line), and 0.79 (dash-dotted line); (b) $n=N/a$ versus $K^2=(L/R)^2$ in arbitrary units; (c) Λ/R versus K in arbitrary units; and (d) Γ/R versus K^3, in arbitrary units, for balls folded from three different papers: carbon (○), biblia (●), and albanene (◇). Straight lines are the best fittings.

string (see [18]). Besides, we found that the statistical distribution of distances between the entry and exit points in the unfolded sheet of size L is best fitted by the inverse Gaussian distribution [19] with the mean $\Lambda \propto KR$ [see Fig. 2(c)] [20], whereas the length of the string path in the unfolded sheet is found to conform to a log-normal distribution [19] with the mean $\Gamma \propto RK^3$ [see Fig. 2(d)]. So the mean distance between intersections of the string with the sheet in the unfolded state is $\ell = \langle \ell_i \rangle = \Gamma/N \propto KR$, whereas in the folded state this distance is proportional to $R/N \propto K^{-2}R$. We also found that the statistical distribution of distances ℓ_i between intersections in the unfolded state is best fitted with the log-normal distribution [19].

To study the scaling properties of the internal structure of randomly folded sheets, the balls were crossed over with the silk strings in such a way that the Euclidean distance (chord length) between the entry and exit points in the folded state r was varied from R to $0.1R$ [see Figs. 1(d)–1(f)]. We found that the number of intersections between the folded sheet and straight line (one-dimensional string) scales as

$$N = aK^2\left(\frac{r}{R}\right)^{D_l-2}, \tag{2}$$

where the local fractal dimension D_l is found to be the same for all folded paper sheets [see Fig. 3(a)]; namely, we found that

$$D_l = 2.68 \pm 0.05 > D. \tag{3}$$

Scaling behavior (2) implies that the local mass density of the folded sheet behaves as

$$\rho = \rho_0\left(\frac{R}{r}\right)^{3-D_l}\left(\frac{R}{h}\right)^{D-3} \propto r^{-(3-D_l)}R^{-(D_l-D)}, \tag{4}$$

where ρ_0 is the material-dependent constant, r^3 is the local volume, and $D < D_l \leq 3$. So in contrast to the case of (statistically) self-similar fractals, for which $D_l = D$ [17], the local mass density within a randomly folded ball depends not only on the size of local volume, but also on the ball diameter. Accordingly, in analogy with the concept of intrinsically anomalous kinetic roughening, which is applied to interfaces characterized by different roughness exponents in the local (ζ) and the global ($a > \zeta$) scales (see [21]), the observed behavior (4) of randomly folded sheets can be termed as the *intrinsically anomalous self-similarity*.

Besides, we found that the mean distance between the entry and exit points in the unfolded sheet satisfies the scaling relation

$$\frac{\Lambda}{L} \propto \left(\frac{r}{R}\right)^{\psi}, \tag{5}$$

with $\psi = 0.3 \pm 0.06$ for all kinds of paper used in this work [see Fig. 3(b)]. We noted that this exponent is in agreement

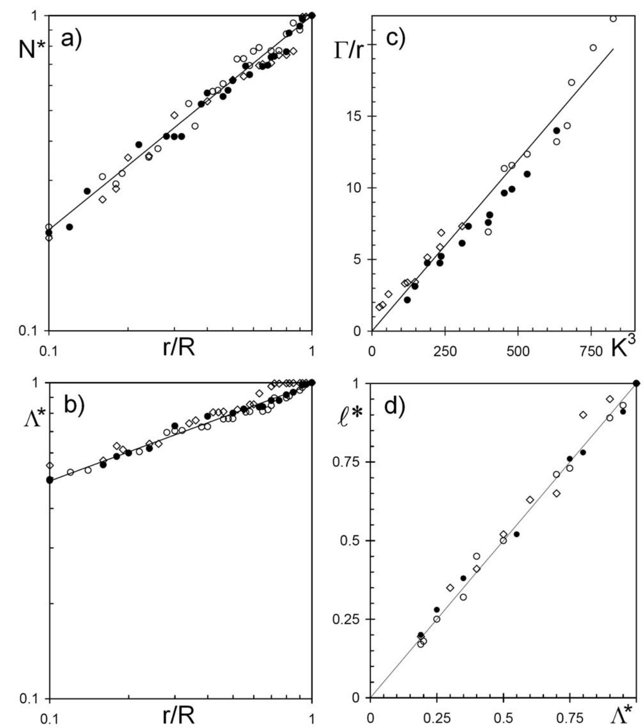

FIG. 3. Log-log plots of (a) $N^*=N/N(r=R)$ versus r/R, where the slope of straight line is 0.68; (b) $\Lambda^*=\Lambda(r)/\Lambda(R)$ versus r/R, where the slope of straight line is 0.3; (c) Γ/r versus K^3; and (d) $\ell^*=\ell/\ell(R)$ versus $\Lambda^*=\Lambda/\Lambda(R)$ in arbitrary units, for balls folded from three different types of paper: carbon (○), biblia (●), and albanene (◇).

with the finding $\Lambda \propto r^{0.33}$ in experiments [16], where the distance Λ was measured as a function of the Euclidean distance r between two points on the surface of balls folded from paper sheets of size $L=660$ mm ($R=43$ mm). However, in contrast to [16], where it was argued that $\psi=D-2$ [22], we state that

$$\psi = 3 - D_l, \tag{6}$$

such that the total length of the string path along the intersection in the unfolded state scales as

$$\Gamma \propto K^3 r, \tag{7}$$

as is shown in Fig. 3(c), whereas the mean distance between intersections in the unfolded state, $\ell=\langle \ell_i \rangle$, is found to conform to a lognormal distribution [16] with the mean proportional to Λ [see Fig. 3(d)]. Indeed, $\Gamma \propto N\ell \propto LK^2(r/R)^{D_l-2+\psi}$ and so from the scaling relation (7) follows the equality (6).

On the other hand, we noted that the local fractal dimension of randomly folded patterns, $D_l=2.68\pm0.05$ ($D_l=3-\psi=-2.7\pm0.06$), coincides with the universal fractal dimension $D=8/3$ expected for randomly folded phantom sheets with finite bending rigidity [23]. This finding implies that the self-avoidance does affect the scaling properties of the internal structure of randomly folded thin matter [24], but plays an important role in the global scaling behavior $R \propto L^{2/D}$. The reason for this is that in sheets with the finite bending rigidity the sheet intersections are restricted by the finite size of crumpling creases, $\Delta \propto L(F/Yh)^{-\delta}$, rather than by the condition of self-avoidance, whereas the global scaling behavior (1) is controlled by the restrictions imposed by the condition of self-avoidance at the global scale $\xi \cong R \gg \Delta$ [25]. Hence, one can expect that intrinsically anomalous self-similarity is an essential feature of the randomly folded matter. Moreover, one may expect this type of anomalous self-similarity may be observed for diverse biological structures, such as cell membranes, tumors, plants (see [26]), etc., where the characteristic size of internal structure may be system size dependent.

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[1] D. P. Francesco and E. Guitter, Phys. Rep. **415**, 1 (2005); M. Venturoli *et al.*, *ibid.* **438**, 1 (2006); G. M. Grason, *ibid.* **433**, 1 (2006).
[2] *Witten's Lectures on Crumpling*, edited by A. J. Wood [Physica A **313**, 83 (2002)]; T. A. Witten, J. Phys.: Condens. Matter **17**, S1651 (2005).
[3] B. E. Hobbs *et al.*, Stress, Strain, Struct. **2**, 9 (2000).
[4] B. A. DiDonna and T. A. Witten, Phys. Rev. Lett. **87**, 206105 (2001); K. Matan, R. B. Williams, T. A. Witten, and S. R. Nagel, *ibid.* **88**, 076101 (2002); E. Cerda and L. Mahadevan, *ibid.* **90**, 074302 (2003); A. Aström, J. Timonen, and M. Karttunen, *ibid.* **93**, 244301 (2004); S. Chaieb, V. K. Natrajan, and A. A. El-rahman, *ibid.* **96**, 078101 (2006).
[5] D. L. Blair and A. Kudrolli, Phys. Rev. Lett. **94**, 166107 (2005).
[6] E. Sultan and A. Boudaoud, Phys. Rev. Lett. **96**, 136103 (2006).
[7] G. A. Vliegenthart and G. Gompper, Nat. Mater. **5**, 216 (2006).
[8] A. E. Lobkovsky, S. Gentges, H. Li, D. Morse, and T. A. Witten, Science **270**, 1482 (1995); E. M. Kramer and T. A. Witten, Phys. Rev. Lett. **78**, 1303 (1997); G. Gompper, Nature (London) **386**, 439 (1997).
[9] A. S. Balankin *et al.*, Phys. Rev. E **73**, 065105(R) (2006).
[10] A. S. Balankin *et al.*, Phys. Rev. E **74**, 061602 (2006).
[11] A. S. Balankin *et al.* Phys. Rev. E **75**, 051117 (2007).
[12] M. J. Bowick and A. Travesset, Phys. Rep. **344**, 255 (2001).
[13] For phantom membranes without bending rigidity $R \propto \ln L$, whereas randomly folded self-avoiding sheets with zero bending rigidity are characterized by $D=2.5$ (see [12]).
[14] M. A. F. Gomes, Am. J. Phys. **55**, 649 (1987); J. Phys. A **23**, L1281 (1990); M. A. F. Gomes and J. H. P. Soares, J. Phys. D **20**, L283 (1987); M. A. F. Gomes *et al.*, *ibid.* **22**, 1217 (1989); R. H. Ko and C. P. Bean, Phys. Teach. **29**, 78 (1991); R. Cassia-Moura and M. A. F. Gomes, J. Theor. Biol. **238**, 331 (2006).

[15] J. B. C. Garcia, M. A. F. Gomes, T. I. Jyh, and T. I. Ren, J. Phys. A **25**, L353 (1992).

[16] M. A. F. Gomes, T. I. Jyh, and T. I. Ren, J. Phys. A **23**, L1281 (1990).

[17] J. Feder, *Fractals* (Plenum, New York, 1988).

[18] S. Miyazima and H. E. Stanley, Phys. Rev. B **35**, 8898 (1987).

[19] The goodness-of-fit tests with the χ^2, Kolmogorov-Smirnov, and Anderson statistics were performed with the help of @RISK software (http://www.palisade.com).

[20] We noted that this scaling relation is consistent with the finding in [14], where it was found that $l \propto r^{1/3} R^{0.86\pm0.06}$ and so $l \propto R^{1.2\pm0.06} \propto L^{0.96\pm0.06}$, when $r=R$. Moreover, the authors of [14] have found that for balls folded from metal strings also $l \propto L^{1.01\pm0.06}$.

[21] J. J. Ramasco, J. M. López, and M. A. Rodríguez, Phys. Rev. Lett. **84**, 2199 (2000).

[22] Notice also the difference between scaling relation (5) and Eq. (1) in Ref. [14].

[23] Moreover, we noted that this finding is in agreement with the universality of local roughness exponent $\zeta=0.72\pm0.04$ for randomly crumpled sheets [10]. Indeed, $D_l \cong 8/3 \cong 2/\zeta$, where the symbol $\cong$ denotes the numerical equality, whereas $D=2/\alpha$, where $\alpha > \zeta$ is the material-dependent global roughness exponent for randomly folded sheets (see [10]).

[24] The self-avoidance would play a determinant role at extremely high confinement $K=L/R$, close to the critical value $K_c = (\pi L/6h)^{1/3}$. For sheets of size $L=100$ cm, $K_c \geq 30$, whereas the confinement of all balls studied in this work is less than 10.

[25] An analogous phenomenon is observed for polymer chains in solvents, the self-avoidance of which becomes significant only at scales larger than thermal blob length, which may coincide with the characteristic chain size [see T. A. Witten, Rev. Mod. Phys. **70**, 1531 (1998)].

[26] *Fractals in Biology and Medicine*, edited by G. A. Losa, D. Merlini, T. F. Nonnenmacher, and E. R. Weibel (Birkhäuser, Berlin, 2002).